The Steenrod algebra action on Dickson algebra generators and Peterson's polynomials

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ABSTRACT. Using Peterson's polynomials, we provide excplicit formulas for the action of the Steenrod algebra on Dickson algebra generators for the mod-odd case.

1. Introduction

The action of the Steenrod algebra on Dickson algebra has been under investigation mainly because it plays an important role in stable homotopy theory and has geometric applications, (see [**2**]). The value of this action on generators has been given for $p = 2$ and partially for p an odd prime.

The main theme of this work is theorem 1 and 2. Those theorems are included in section three, where we investigate the value of the action on the two extreme generators (with respect to degree) of Dickson algebra. Combining theorems 1 and 2 with well known theorems (see [**3**]), the action can be calculated for any generator.

The key point is this action on Peterson's polynomials. We are interested in a special class of Peterson's polynomials called leading Peterson's polynomials. The size of this set is given by a Fibonacci sequence.

In section two, well known results are recollected from the literature for completeness.

2. Dickson and Symmetric invariants

Let $V^{(i)}$ be the i-th dimensional vector space over the field F_p of p-elements generated by $\{y_1, ..., y_i\}$ and GL_i the general linear group acting as usual. Let also GL_n act on the polynomial algebra $P_n := F_p[y_1, ..., y_n]$ by the induced action. P_n is graded by $|y_t| = 2$ (for topological reasons).

Since $P_n \leq H^*(V^{(n)}, F_p)$, the Steenrod algebra acts naturally on.

Let h_i be the polynomial given by .

$$h_i = \prod_{a \in V^{(i-1)}} (y_i + a) \tag{2.1}$$

which has degree $2p^{i-1}$. Let us note that $y_i^{p^{i-1}}$ is a summand in h_i and the last polynomial is invariant under the upper triangular group U_n where only one's are

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allowed on the main diagonal. This is because $g(y_i + a) = y_i + b + a'$, where $b, a' \in V^{(i-1)}$. If non-zero elements are allowed on the diagonal, then h_i^{p-1} is invariant under the Borel subgroup B_n:

$$g(y_i + a)^{p-1} = (cy_i + b + a')^{p-1} = c^{p-1}(y_i + b' + a'')^{p-1}.$$

Since the set $\{h_1^{p-1}, ..., h_n^{p-1}\}$ is algebraically independent and its elements have the right degrees with respect to the order of the group B_n, the corresponding ring of invariants $P_n^{B_n}$ is a polynomial algebra ([5]): $F_p[h_1^{p-1}, ..., h_n^{p-1}]$.

The following proposition is known:

PROPOSITION 1. *Let $f \in P_n$, then it is a GL_n-invariant iff f is symmetric and invariant under the transformation $y_t \to y_t + cy_i$, and $y_i \to y_i$ for $i = 1, ..., t-1$ and $c \in F_p$.*

Let the symmetric group Σ_n act on P_n by permuting variables, and

$$P_n' := F_p[y_1^{p-1}, y_2^{p(p-1)}, ..., y_n^{p^{n-1}(p-1)}]$$

abbreviate the extended polynomial algebra. Then S_n' is called the extended symmetric algebra where:

$$S_n' := F_p[y_1^{p-1}, y_2^{p(p-1)}, ..., y_n^{p^{n-1}(p-1)}]^{\Sigma_n}$$

which is:

$$F_p[\sum_i y_i^{p^{n-1}(p-1)}, \sum_{i,j} \prod_{i \neq j} (y_j y_i)^{p^{n-1}(p-1)}, \cdots, \prod_i (y_i)^{p^{n-1}(p-1)})]$$

Let Φ_n be the algebra map between P_n' and $P_n^{B_n}$ given by $\Phi_n\left(y_i^{p^{i-1}(p-1)}\right) = h_i^{p-1}$. This is an algebra isomorphism but not a Steenrod algebra map. Then $\Phi_n(S_n') = P_n^{GL_n}$, the so called Dickson algebra, abbreviated by D_n.

Let the generators for the Dickson algebra be $\{d_{n,0}, ..., d_{n,n-1}\}$. Then because of the isomorphism above, the following relations are deduced:

$$d_{n,n-i} = \sum_{1 \le j_1 < \cdots < j_i \le n} \prod_{s=1}^{i} \left(h_{j_s}^{p-1}\right)^{p^{n-i+s-j_s}} \tag{2.2}$$

Moreover, using 2.1 and 2.2 we deduce the following well known formula:

$$h_i = \sum_{t=0}^{i-1} (-1)^{i-1-t} y_i^{p^t} d_{i-1,t} \tag{2.3}$$

Here $d_{i-1,i-1} := 1$. Let us also note that the last two formulas will be of great importance in the sequel.

3. The Steenrod algebra action on Dickson invariants

The action mentioned above has been given for $p = 2$ in [1] and [3]. For p odd, it has been calculated only for the Steenrod algebra generators, P^{p^i}, and particular cases ([3]). Extending the idea used in the previous section, we compute it for any element P^k on the two extreme generators of D_n, namely $d_{n,n-1}$ and $d_{n,0}$.

Let us start with $P^k d_{n,n-1}$. We recall that

$$P^k y_i^n = \binom{n}{k} y^{n+k(p-1)} \tag{3.1}$$

In particular,

$$P^k y_i^{p^n} = \begin{cases} y_i^{p^n} & \text{if } k = 0 \\ y_i^{p^{n+1}} & \text{if } k = p^n \\ 0 & \text{otherwise} \end{cases}$$

Let us define an ordering between sequences $I = (i_n, ..., i_1)$ and $J = (j_n, ..., j_1)$ such that $I > J$ iff $i_t > j_t$ and t is the biggest index with this property; otherwise $I = J$. Next we consider sequences between exponents of various monomials in P_n. The biggest exponent in $d_{n,n-1} = \left(h_n^{p-1} + h_{n-1}^{p(p-1)} + ... + h_1^{p^{n-1}(p-1)}\right)$ is $(p^{n-1}(p-1), 0, ..., 0)$ which is associated with the monomial $y_n^{p^{n-1}(p-1)}$. Let us consider a typical monomial in D_n, $d_{n,0}^{a_0} \cdots d_{n,n-1}^{a_{n-1}}$, its biggest sequence is the following:

$$\left(p^{n-1}(p-1)\left(\sum_0^{n-1} a_t\right), p^{n-2}(p-1)\left(\sum_0^{n-2} a_t\right), \cdots, (p-1)a_0\right) \tag{3.2}$$

We are interested in those natural numbers k such that $P^k d_{n,n-1} = c d_{n,0}^{a_0} \cdots d_{n,n-1}^{a_{n-1}}$ for $c \neq 0$. For degree reasons: $k = p^{n-1}\left(\sum_0^{n-1} a_t - 1\right) + p^{n-2}\left(\sum_0^{n-2} a_t\right) + \cdots + a_0$ and $\sum_0^{n-1} a_t \leq p - 1$. Let $k = \sum_1^n k_t$ and P^{k_t} applies to $y_t^{m_t}$. Let $A_s = a_0 + \cdots + a_s$ and b_t a non-negative integer such that $0 \leq b_t \leq \min[(p-1)b_{t+1}, A_{t-1}]$. Here $s = 0, \cdots, n-1$ and $t = 1, \cdots, n$.

EXAMPLE 1. *Let $n = 3$. We are looking for a k and a c such that $P^k d_{3,2} = P^k\left(h_3^{p-1} + h_2^{p(p-1)} + h_1^{p^2(p-1)}\right) = c d_{3,0}^{a_0} d_{3,1}^{a_1} d_{3,2}^{a_2}$. Let us concentrate only on*

$$P^k\left(h_3^{(p-1)}\right) = P^k\left(y_3^{p^2} - y_3^p d_{2,1} + y_3 d_{2,0}\right)^{(p-1)} \tag{3.3}$$

Of course, $k = k_1 + k_2 + k_3 = p^2(a_0 + a_1 + a_2 - d) + p(a_0 + a_1) + a_0$. Let a summand in 3.3 be $P^k\left((-1)^{b_{3,1}}\binom{(p-1)}{b_{3,1}+b_{3,0}}\binom{b_{3,1}+b_{3,0}}{b_{3,1}} y_3^{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}} d_{2,1}^{b_{3,1}} d_{2,0}^{b_{3,0}}\right) = (-1)^{b_{3,1}}\binom{(p-1)}{b_{3,1}+b_{3,0}}\binom{b_{3,1}+b_{3,0}}{b_{3,1}} P^{k_3} y_3^{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}} P^{k_2} d_{2,1}^{b_{3,1}} P^{k_1} d_{2,0}^{b_{3,0}}$. Next, we consider such a k_3 such that $P^{k_3} y_3^{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}} = \binom{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}}{p^2(a_0+a_1+a_2-1)+p(b_{3,1}+b_{3,0})+b_{3,0}} y_3^{(p-1)p^2(a_0+a_1+a_2)}$ This is identically non zero only if $b_{3,0} = 0$. Hence $k_3 = p^2(a_0 + a_1 + a_2 - 1) + pb_{3,1}$ and the summation runs over $0 \leq b_{3,1} \leq p - (a_0 + a_1 + a_2)$. But $k - k_3 = p(a_0 + a_1 - b_{3,1}) + a_0$ implies that $b_{3,1} \leq a_0 + a_1$. Thus $0 \leq b_{3,1} \leq \min[p - (a_0 + a_1 + a_2), a_0 + a_1]$. Next we consider $P^{k_2+k_1} d_{2,1}^{b_{3,1}} = P^{k_2+k_1}(h_2^{p-1} + h_1^{p(p-1)})^{b_{3,1}}$. The only eligible summand for our target is $P^{k_2+k_1}(h_2^{p-1})^{b_{3,1}}$ (otherwise, the exponent of y_1 exceeds the required one). $P^{k_2+k_1}(y_2^p - y_2 y_1^{(p-1)})^{(p-1)b_{3,1}}$. As before: $P^{k_2} y_2^{p((p-1)b_{3,1}-b_2)+b_2} P^{k_1} y_1^{(p-1)b_2} = \binom{p((p-1)b_{3,1}-b_2)+b_2}{p(a_0+a_1-b_{3,1})+b_2}\binom{(p-1)b_2}{a_0-b_2} y_2^{(p-1)p(a_0+a_1)} y_1^{(p-1)a_0}$ Hence $k_2 = p(a_0 + a_1 - b_{3,1}) + b_2$ and $k_1 = a_0 - b_2$. And the summation runs over $0 \leq b_2 \leq \min((p-1)b_{3,1}, a_0)$. Finally the coefficient of the required element is given by the following sum:

$$\sum_{b_{3,1},b_2} (-1)^{b_{3,1}+b_2}\binom{(p-1)}{b_{3,1}}\binom{(p-1)b_{3,1}}{b_2}\binom{(p-1)-b_{3,1}}{a_0+a_1+a_2-1}\binom{p((p-1)b_{3,1}-b_2)+b_2}{p(a_0+a_1-b_{3,1})+b_2}\binom{(p-1)b_2}{a_0-b_2}$$

THEOREM 1. *Let* $k = p^{n-1}\left(\sum_0^{n-1} a_t - 1\right) + p^{n-2}\left(\sum_0^{n-2} a_t\right) + \cdots + a_0$ *with* $\sum_0^{n-1} a_t \leq p-1$. *Then* $P^k d_{n,n-1} = c d_{n,0}^{a_0} \cdots d_{n,n-1}^{a_{n-1}}$, *where c is the following constant* mod $-p$.
$$\sum_{b_t}(-1)^b \binom{p-1}{b_n}\binom{(p-1)b_n}{b_{n-1}} \cdots \binom{(p-1)b_3}{b_2}\binom{p-1-b_n}{A_{n-1}-1}\binom{(p-1)b_n-b_{n-1}}{A_{n-2}-b_n} \cdots \binom{(p-1)b_t-b_{t-1}}{A_{t-2}-b_t} \cdots \binom{(p-1)b_2}{A_0-b_2}$$

Here $b = \sum_1^{n-1} b_t$.

PROOF. It suffices to consider $P^k(h_n^{(p-1)})$. We prove the corresponding formula for $P^k(h_n^{(p-1)d})$ by induction on n. Here $k = p^{n-1}\left(\sum_0^{n-1} a_t - d\right) + p^{n-2}\left(\sum_0^{n-2} a_t\right) + \cdots + a_0$, $1 \leq d \leq p-1$ and $\sum_0^{n-1} a_t \leq p-1$. The case $n=3$ has been worked out in the last example. Let us recall that we are looking for the coefficient of the monomial $\prod_{i=1}^{n} y_i^{(p-1)p^{i-1}\left(\sum_0^{i-1} a_t\right)}$ after applying P^k. Expanding $(h_n)^{(p-1)d} = \left(\sum_{t=0}^{n-1}(-1)^{n-1-t} y_n^{p^t} d_{n-1,t}\right)^{(p-1)d}$ and considering the coefficient of

$$P^{k_n} y_n^{p^{n-1}[(p-1)d-b_{n,i_1}-\cdots-b_{n,i_{p-1}}]+p^{i_1-1}b_{n,i_1}+\cdots+p^{i_{p-1}-1}b_{n,i_{p-1}}} \tag{3.4}$$

We conclude that only $b_n := b_{n,n-1} \neq 0$. Moreover, $0 \leq b_{n,n-1} \leq (p-1)d$. Thus we proceed to the following element:

$$\sum_{b_n}(-1)^{b_n}\binom{(p-1)d}{b_n}\binom{(p-1)d-b_n}{A_{n-1}-1} y_n^{(p-1)p^{n-1}\left(\sum_0^{n-1} a_t\right)} P^{k-k_n} d_{n-1,n-2}^{b_n} \tag{3.5}$$

Here $k - k_n = p^{n-2}\left(\sum_0^{n-2} a_t - b_n\right) + p^{n-3}\left(\sum_0^{n-3} a_t\right) + \cdots + a_0$. ∎

Let us proceed to the case $P^k d_{n,0} = c\prod_{t=0}^{n-1} d_{n,t}^{a_t}$. Restrictions imply that $k = \sum_0^{n-1}(a_0 + \cdots + a_{t-1})p^t \leq p^n - 1$ and $a_0 \geq 1$. The biggest exponents in $d_{n,0}$ and $c\prod_{t=0}^{n-1} d_{n,t}^{a_t}$ are $(p^{n-1}(p-1), \cdots, (p-1))$ and $(p^{n-1}(p-1)(a_0 + \cdots + a_{n-1}), \cdots (p-1)a_0)$ respectively. The idea is to consider all monomials f in $d_{n,0} = \prod h_i^{p-1} = \left(\sum_\sigma [y]^{p^\sigma}\right)^{p-1}$ such that $P^k f = c_f [y]^{(p^{n-1}(p-1)(a_0+\cdots+a_{n-1}), \cdots (p-1)a_0)}$ and c_f non-identically zero. Here σ is a permutation on $\{0, \cdots, n-1\}$ and $[y]^{p^\sigma} = y_n^{p^{\sigma_0}} \cdots y_1^{p^{\sigma_{n-1}}}$. All coefficients c_f are added and that constant will be the coefficient c of $\prod_{t=0}^{n-1} d_{n,t}^{a_t}$ in $P^k d_{n,0}$. Note that this decomposition holds for this particular generator only. Let us recall that an element of the form $[y]^{p^\sigma}$ is called a Peterson polynomial and $\prod h_i$ contains all such polynomials of the given degree $1 + \cdots + p^{n-1}$. Among those, we consider only the ones with the right degree called **leading Peterson's polynomials**. Our first task is to find the leading Peterson's polynomials. The cardinality of this set of polynomials is given by the n-th element of a Fibonacci sequence.

PROPOSITION 2. *Let $[y]^{p^I} = y_n^{p^{i_n}} \cdots y_1^{p^{i_1}}$, for $0 \le i_t \le n-1$. If $t < i_t$ or $i_t < t-2$, then $P^k[y]^{p^I}$ does not contain a multiple of $[y]^{(p^{n-1}(p-1)(a_0+\cdots+a_{n-1}),\cdots(p-1)a_0)}$ where $k = \sum_0^{n-1} (a_0 + \cdots + a_{t-1})\, p^t$.*

PROOF. Let us recall that $d_{n,0} = \left(\sum_\sigma [y]^{p^\sigma}\right)^{p-1} = \sum c_{(I_1,\cdots,I_{p-1})} \prod_{s=1}^{p-1} [y]^{p^{I_s}}$. Here $I_s = (i_{n,s}, \cdots, i_{1,s})$. A Steenrod operation acts on monomials by Cartan formula: $k = k_n + \cdots + k_1$. Let us consider $P^{k_t} y_t^{\sum p^{i_{s,t}} e_s}$ a typical summand in $d_{n,0}$. Here $1 \le e_s \le p-1$.

(3.6) $\quad \sum p^{i_{s,t}} e_s + k_t(p-1) = p^{t-1}(p-1)(a_0 + \cdots + a_{t-1}) \Longrightarrow i_{s,t} \le t$

Let $\sum p^{i_{s,t}} e_s = p^t E_t + p^{t-1} E_{t-1} + \cdots + pE_1 + (p-1-E_t - \cdots - E_1)$, then

(3.7)

$k_t = p^{t-1}(a_0 + \cdots + a_{t-1} - E_t - 1) + p^{t-2}(p-1-E_t - E_{t-1}) + \cdots + (p-1-E_t - \cdots - E_1)$

Hence Steenrod's binomial coefficients are as follows:

$$(3.8) \qquad \binom{E_{t-1}}{a_0 + \cdots + a_{t-1} - E_t - 1} \binom{E_{t-2}}{p-1-E_t-E_{t-1}} \cdots \binom{E_1}{p-1-E_t - \cdots E_2}$$

The claimed restrictions are induced: $E_t + E_{t-1} + 1 \ge a_0 + \cdots + a_{t-1}$ and $a_0 + \cdots + a_{t-1} \ge E_t + 1$. $p-1 = E_t + E_{t-1} + E_{t-2} \Longrightarrow E_{t-s} = 0$ for $s > 2$. ∎

DEFINITION 1. *A Peterson polynomial $y_n^{p^{i_n}} \cdots y_1^{p^{i_1}}$ satisfying $n-2 \le i_n \le n-1$, $t-2 \le i_t \le t$ and $0 \le i_1 \le 1$ is called a leading Peterson polynomial. The set of leading Peterson polynomials is denoted by LPP_n.*

LEMMA 1. *The size of LPP_n is given by the n-th element of a Fibonacci sequence.*

PROOF. Let $n = 2$, then there are only two pairs of exponents: $(1,0)$ and $(0,1)$, $F_2 = 2$. Let $n = 3$, then there are only three triples of exponents: $(2,1,0)$, $(2,0,1)$ and $(1,2,0)$, $F_3 = 3$. Given F_k for $k < n$, F_n counts sequences of the form $(n-1, i_{n-1}, \cdots, i_1)$ plus $(n-2, n-1, i_{n-2}, \cdots, i_1)$. The size of the first set is F_{n-1} and the second F_{n-2}. ∎

Our problem reduces to the case $P^k(LPP_n)^{p-1}$.

THEOREM 2. *Let $k = \sum_0^{n-1} (a_0 + \cdots + a_{t-1})\, p^t$, then $P^k d_{n,0} = c \prod_{t=0}^{n-1} d_{n,t}^{a_t}$ where the coefficient c is given by:*

$$(3.9) \qquad \left[\sum_{\substack{I_t \in ELPP_n \\ E_1 + \cdots E_{p-1} = p-1}} \frac{(p-1)!}{E_1! \cdots E_{p-1}!} \prod_{t=1}^{n} \binom{B_{t-1,t}}{a_0 + \cdots + a_{t-1} - 1 - B_{t,t}} \right] \bmod p$$

Here $ELPP_n$ is the set which contains all exponents of monomials from LPP_n and $B_{t,s}$ is defined inductively as follows:

$B_{n-1,n} = \sum E_s(i_{n,s} - (n-2))$;

$B_{n-2,n} = p - 1 - B_{n-1,n}$;
$B_{t-1,t-1} = p - 1 - B_{t-1,t} - B_{t-1,t+1}$;
$B_{t-2,t-1} = \sum E_s(i_{t-1,s} - (t-3)) - B_{t-1,t-1}$;
$B_{t-3,t-1} = p - 1 - B_{t-1,t-1} - B_{t-2,t-1}$.

PROOF. Since $k < p^n$, the value of P^k on $d_{n,0}$ is a monomial. Last proposition implies that all coefficients in $P^k(LPP_n)^{p-1}$ must be added up. It remains to define the terms $B_{t,s}$. Let us recall that we are considering monomials of the form $\prod_t (y_n^{p^{i_{n,t}}} \cdots y_1^{p^{i_{1,t}}})^{E_t}$. For each variable, the coefficients of powers of its exponents add up to $p-1$. Let us consider y_n: $B_{n-1,n}p^{n-1} + B_{n-2,n}p^{n-2} = \sum p^{i_{n,s}} E_s$. Then $B_{n-1,n} = \sum E_s(i_{n,s} - (n-2))$ and $B_{n-2,n} = p - 1 - B_{n-1,n}$. For y_{n-1}, we have the following equation $B_{n-1,n-1}p^{n-1} + B_{n-2,n-1}p^{n-2} + B_{n-3,n-1}p^{n-3} = \sum p^{i_{n-1s}} E_s$ which implies $B_{n-1,n-1} = p - 1 - B_{n-1,n}$, $B_{n-2,n-1} = \sum E_s(i_{n-1,s} - (n-3)) - B_{n-1,n-1}$ and $B_{n-3,n-1} = p - 1 - B_{n-2,n-1} - B_{n-1,n-1}$. Now the claimed formulas for $B_{t,s}$ are easily deduced. ∎

Let us recall the analogue formula for $P^k(h_n)^{p-1}$ from theorem 10 page 950 in **[3]**.

THEOREM 3. **[3]**
(3.10)

$$P^k(h_n)^{p-1} = \begin{cases} \frac{1}{d_{n-1,0}}\left(P^k d_{n,0} - {h_n}^{p-1}(P^k d_{n-1,0}) + \sum_{m=0}^{n-2} d_{n-1,n-2-m} P^{k-p_m} d_{n-1,0}\right) \\ 0, \ if \ \ k \neq \sum_{m=0}^{n-1} c_m p_m \end{cases}$$

Here $p_m = p^{n-1} + \cdots + p^{n-1-m}$.

Using formula 2.2, Cartan formula, and the two last theorems, the interested reader can evaluate $P^k d_{n,s}$ for $0 < s < n-1$.

References

[1] N. H. V. Hung, *The action of Steenrod squares on the modular invariants of linear groups*. Proc. Amer. Soc., 113, 1991, 1097-1104.

[2] N. H. V. Hung and F. P. Peterson, *Spherical classes and the Dickson algebra*, Math. Proc. Camb. Phil. Soc., 125 (1998), 253-264.

[3] Nondas E. Kechagias, *The Steenrod algebra action on generators of subgroups of $GL_n(Z/pZ)$*. Proc. Amer. Soc., 118, 1993. 943-952.

[4] Huyhn Mui. *Modular invariant theory and the cohomology algebras of the symmetric groups*. J. Fac. Sci. Univ. Tokyo. **IA** (1975), 319-369.

[5] C. Wilkerson. *A primer on the Dickson invariants*, Contemp. Math., vol. 19, Amer. Math. Soc., 1983, 421-434.

[6] R. M. W. Wood, *Problems in the Steenrod Algebra*, Bull. London Math. Soc. 30 (1998), 449-517.

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